where now all of the Lagrangian time derivatives in Eq. (4) are evaluated with respect to the vortex velocity; i.e., $d(\ )/dt = \partial(\ )/\partial t + \frac{1}{2}(V_1 + V_2)\cdot\nabla(\ )$ for each derivative in Eq. (4). Thus the derivatives on the right-hand side of Eq. (4) are different from the derivatives on the left-hand sides of Eqs. (3), and Eqs. (3) may not be substituted directly into Eq. (4), as was done in Ref. 1.

By properly accounting for the differences in the Lagrangian derivatives following the vortex in Eq. (4) and the Lagrangian derivatives following the fluid particles in Eqs. (3), the final equation for the rate of change of circulation becomes

$$\frac{d\Gamma_i}{dt} = 2\kappa\Delta s_i\left[g\sin\theta_i + \left(\frac{dV_i}{dt}\right)_s + \frac{1}{4}\gamma_i\left(\frac{\partial\gamma}{\partial s}\right)_{s=s_i}\right] - \frac{2\sigma}{\rho_1+\rho_2}\Delta s_i\left(\frac{\partial R^{-1}}{\partial s}\right)_{s=s_i} \tag{5}$$

where $\kappa \equiv (\rho_2-\rho_1)/(\rho_1+\rho_2)$, and $\gamma_i = (V_{1s} - V_{2s})_{s=si}$ is the circulation per unit arc length [the approximation $\gamma_i = \Gamma_i/\Delta s_i$ can be used for numerical calculations, with an appropriate finite-differencing for the $(\partial\gamma/\partial s)_{s=si}$ factor]. Equation (5) differs from Eq. (14) of Ref. 1 by the addition of the $\frac{1}{4}\gamma_i(\partial\gamma/\partial s)_{s=s_i}$ term in the brackets. Equation (5) is consistent with the equation given by Zaroodny and Greenberg[2] for a continuous vortex sheet.

In order to determine the magnitude of the error incurred by using Eq. (14) of Ref. 1, we now consider the linearized case corresponding to the classical Kelvin-Helmholtz problem. For a coordinate system at rest with respect to an unperturbed vortex sheet along the x direction, linearization is achieved by first setting $\gamma = \gamma_0 + \gamma'$, $V_x = V_x'$, $V_y = V_y' = \partial\eta/\partial t$, with γ being the circulation per unit length, γ_0 is the unperturbed circulation per unit length (which is equal to the unperturbed velocity change ΔU across the sheet), and η is the interface displacement. After linearization, the continuous form of Eq. (5) [Eq. (12) of Ref. 2] and the Biot-Savart integrals for the vortex velocity components become

$$\frac{\partial\gamma'}{\partial t} = -\gamma_0\frac{\partial V_x'}{\partial x} + 2\kappa\left[g\frac{\partial\eta}{\partial x} + \frac{\partial V_x'}{\partial t} + \frac{1}{4}\gamma_0\frac{\partial\gamma'}{\partial x}\right] - \frac{2\sigma}{\rho_1+\rho_2}\frac{\partial^3\eta}{\partial x^3} \tag{6a}$$

$$V_x' = \frac{\gamma_0}{2\pi}\int_{-\infty}^{\infty}\frac{[\eta(x,t)-\eta(\xi,t)]}{(x-\xi)^2}d\xi \tag{6b}$$

$$\frac{\partial\eta}{\partial t} = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma'(\xi,t)}{(x-\xi)}d\xi \tag{6c}$$

where the integrals are determined by their Cauchy principal values.

If γ', V_x', and η are assumed to be proportional to $\exp(ikx+nt)$, then Eqs. (6) are solved by the following value of n;

$$n = i\frac{\kappa}{2}\Delta Uk \pm\left[\frac{1}{4}(1-\kappa^2)(\Delta U)^2k^2 - \kappa gk - \frac{\sigma k^3}{\rho_1+\rho_2}\right]^{\frac{1}{2}} \tag{7}$$

Equation (7) is the classical Kelvin-Helmholtz result, and is given as Eq. (1) in Ref. 1. If Eq. (14) of Ref. 1 is used for the rate of change of circulation, then the linearized calculation gives an expression for n of the same form as Eq. (7), but with κ replaced by $(\frac{1}{2})\kappa$ in the first term on the right, and κ^2 by $(\frac{1}{4})\kappa^2$ in the first term under the square root.

Although Zalosh's numerical calculations apparently were matched to the correct linear Kelvin-Helmholtz formula at the initial time, errors in his use of his Eq. (14) should be comparable to the differences in the two linear values for n. For the small density differences considered by Zalosh, $\kappa < 0.053$, the two equations for n give little difference in the periods of amplitude oscillation for stable conditions. The two formulas for n give very small differences in the amplitude growth rates for the unstable conditions considered by Zalosh, and the phase velocities are small (but differ by a factor of two). Thus, we would expect his numerical results to be reasonably accurate. On the other hand, if Eq. (14) of Ref. 1 is used for large density differences, i.e., κ not small, then large errors result. Note that, in the single-fluid limit of $\rho_1 \to 0$, i.e., $\kappa \to 1$, Eq. (7) shows that the shear instability term $(\frac{1}{4})(1-\kappa^2)(\Delta U)^2$ disappears. For the incorrect formula derived from Eq. (14) of Ref. 1, the shear instability would remain in the limit $\kappa \to 1$.

References

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THE comments on the missing term in the circulation equation are painfully acknowledged. In general, the term

$$1/4\ \gamma_i(\partial\gamma/\partial s)_{s=s_i}$$

should appear in the brackets in Eq. (14) of Ref. 1, and the term

$$\frac{1}{4}\frac{(1-S)}{(1+S)}\gamma_i(\gamma_{i+1}-\gamma_{i-1})$$

should appear on the right-hand side of Eq. (16c).

Fortunately, the addition of the missing term should not alter significantly the calculated results presented in Ref. 1. This can be seen by evaluating the ratio of the missing term to the gravitational acceleration term, i.e., the first term on the right-hand side of Eqs. (14) and (16c). For the initial sinusoidally perturbed vortex sheet, this ratio (rms averaged over one wavelength) is equal to

$$\left|\overline{\frac{\frac{\Delta U}{4}\frac{\partial\gamma}{\partial s}}{g\frac{\partial\eta}{\partial s}}}\right| = \frac{\pi}{2}\frac{(\Delta U)^2}{g\lambda}$$

Most of the calculated results presented in Ref. 1 were obtained for small values of the ratio $\pi(\Delta U)^2/2g\lambda$. For example, the curves shown in Fig. 6 of Ref. 1 ($F_r^{-2}=1$, $S=0.9$) correspond to a ratio of 0.083. The only case for which this ratio is not much smaller than unity is the situation depicted in Fig. 7 of Ref. 1 ($F_r^{-2}=0.10$, $S=0.90$). In that case, the 7% discrepancy between the calculated period of oscillation and the results of linear theory, may be due partially to the inadvertently omitted term in the circulation equation.

Reference

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Received April 18, 1977.
Index categories: Hydrodynamics; Subsonic Flow; Transonic Flow.

*Presently with Factory Mutual Research Corporation, Norwood, Mass. Member AIAA.